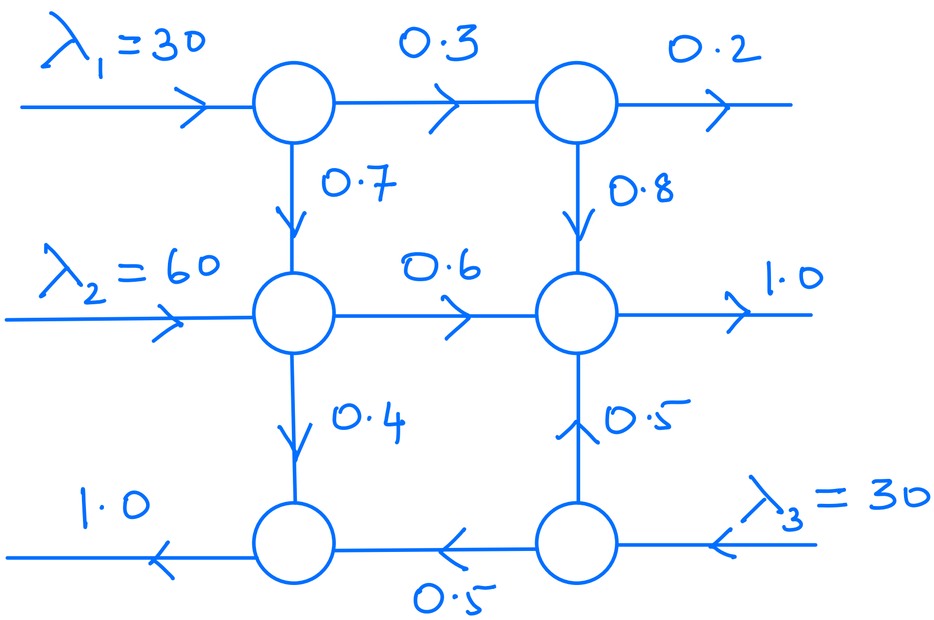
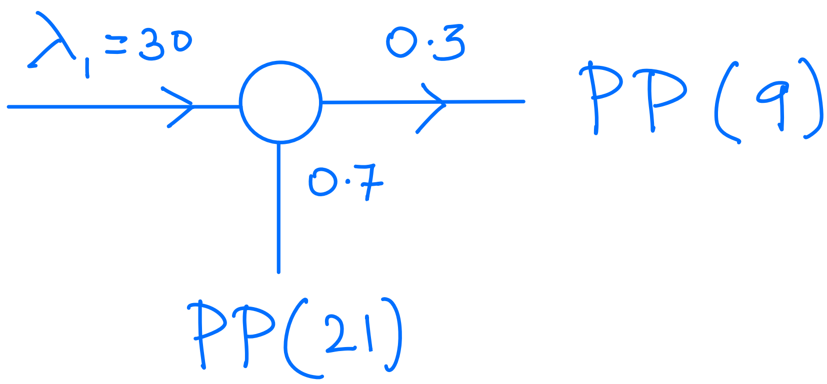
ASSIGNMENT III

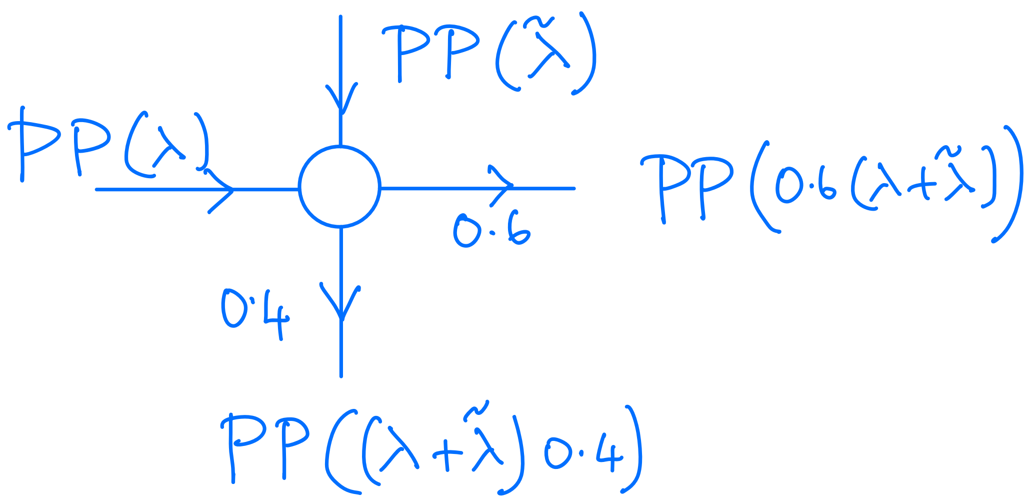
1. Consider the network below. Inputs are Poisson processes with indicated rates, and the numbers are probabilites of vehicles choosing the indicated direction.



Describe the traffic flow on each branch.

Solution Sketch





Why is the above true?

Use the following two observations.

(i) If $\{X_t, t \geq 0\}$ is a Poisson process with rate λ and $\{\tilde{X}_t, t \geq 0\}$ is a Poisson process with rate $\tilde{\lambda}$, then $\{X_t + \tilde{X}_t, t \geq 0\}$ is Poisson with rate $\lambda + \tilde{\lambda}$.

(ii) Let $\{X_t, t \geq 0\}$ be a Poisson process with rate λ. Let $\{B_n, n = 1, 2, \ldots\}$ be a Bernoulli process (sequence of iid Bernoullis) with success probability $p > 0$. Suppose $\{X_t, t \geq 0\}$ and $\{B_t, t \geq 0\}$ are independent.

Then the process

V_t = number of successes by time t

is a Poisson process with rate λp.

2. Let Y be a random variable taking values

$$E = \{a, b, \ldots\}$$

with distribution

$$\pi = (\pi(a), \pi(b), \ldots)$$

For each $i \in E$, suppose $\{X_{t,i}, t \geq 0\}$ are Poisson processes independent of each other and of Y.

Define

$$X_t = X_{t,Y}.$$

(a) Find $P(X_t = k)$.

(b) Show that in general $\{X_t, t \geq 0\}$ does not have independent increments.

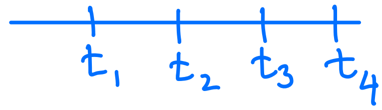
Solution Sketch

(a) $P(X_{t,Y} = k)$

$$= \sum_{y \in E} P(X_{t,Y} = k \mid Y = y)\, P(Y = y)$$

$$= \sum_{y \in E} P(X_{t,y} = k)\, P(Y = y)$$

(ind. of $X_{t,i}$ with Y)



(b) $$P\left(X_{t_2,Y} - X_{t_1,Y} \in A,\; X_{t_4,Y} - X_{t_3,Y} \in B\right)$$

$$= \sum_{y \in E} P\left(X_{t_2,Y} - X_{t_1,Y} \in A,\; X_{t_4,Y} - X_{t_3,Y} \in B \,\middle|\, Y = y\right) P(Y = y)$$

$$= \sum_{y \in E} P\left(X_{t_2,y} - X_{t_1,y} \in A,\; X_{t_4,y} - X_{t_3,y} \in B\right) P(Y = y)$$

(ind. of $Y_{t,y}$ with Y)

$$= \sum_{y \in E} P\left(X_{t_2,y} - X_{t_1,y} \in A\right) \times P\left(X_{t_4,y} - X_{t_3,y} \in B\right) \times P(Y=y)$$

$$\neq \left(\sum_{y \in E} P\left(X_{t_2,y} - X_{t_1,y} \in A\right) P(Y=y) \right)$$

$$\times \left(\sum_{y \in E} P\left(X_{t_2,y} - X_{t_1,y} \in A\right) P(Y=y) \right)$$

$\blacksquare$

3. Suppose $\{X_t, t \geq 0\}$ has independent and stationary increments, and that

(i) $f(t) = \mathbb{E}[X_t] < \infty$

(ii) f is differentiable.

Show that

$$f(t) = m_0 + m_1 t$$

where $m_0 = f(0)$, $m_1 = f(1) - f(0)$.

Solution Sketch

$$f(s+t) = \mathbb{E}[X_{s+t}]$$

$$= \mathbb{E}[X_s + X_{s+t} - X_s]$$

$$= f(s) + \mathbb{E}[X_{s+t} - X_s]$$

$$\overset{\text{st. inc.}}{=\!=} f(s) + \mathbb{E}[X_t - X_0]$$

$$= f(s) + f(t) - f(0)$$

Differentiate w.r.t t to see that

$$f'(s+t) = f'(t) \quad \forall s, t \geq 0.$$

$$\Rightarrow f'(x) = \text{constant}$$

$$\Rightarrow f(t) = m_1 t + m_0 , \quad t \geq 0.$$

Set $t=0$ to see that $m_0 = f(0)$.

Set $t=1$ to see that $m_1 = f(1) - f(0)$.

$\blacksquare$

4. Let $\{X_t, t \geq 0\}$ be a stationary Poisson process.

An arrival from this process at time $s \in [0, t]$ is classified as Type I with probability $p(s)$ and Type II with probability $1 - p(s)$.

Let $\{Y_t, t \geq 0\}$ and $\{Z_t, t \geq 0\}$ represent arrivals corresponding to the Type I and Type II processes. Show that $\{Y_t, t \geq 0\}$ and $\{Z_t, t \geq 0\}$ are Poisson with rate λq and $\lambda(1-q)$ where

$$q = \frac{1}{t} \int_0^t p(s)\, ds.$$

Solution Sketch

Let $\{Y_t,\ t \geq 0\}$ correspond to the type I arrival process.

Notice:

$$P(Y_t = n) =$$

$$\sum_{m=n}^{\infty} P(Y_t = n \mid X_t = m)\, P(X_t = m)$$

We know that

$$(S_1, S_2, \ldots, S_{X_t}) \mid X_t = m$$

$$\stackrel{d}{=} (U_{(1)}, U_{(2)}, \ldots, U_{(m)})$$

where $U_1, U_2, \ldots, U_m \stackrel{iid}{\sim} \text{Unif}(0, t)$.

$$P(Y_t = n \mid X_t = m)$$

$$= \binom{m}{n} \left(\overset{q}{\int_0^t p(s) \frac{1}{t} \, ds} \right)^n \left(1 - \int_0^t p(s) \frac{1}{t} \, ds \right)^{m-n}$$

Therefore,

$$P(Y_t = n) = \sum_{m=n}^{\infty} \binom{m}{n} q^n (1-q)^{m-n} \frac{e^{-\lambda t}(\lambda t)^m}{m!}$$

$$= e^{-\lambda t} \sum_{m=n}^{\infty} \frac{1}{n!(m-n)!} (\lambda q t)^n (\lambda(1-q)t)^{m-n}$$

$$= \frac{e^{-\lambda t}(\lambda q t)^n}{n!} \sum_{k=0}^{\infty} \frac{(\lambda(1-q)t)^k}{k!}$$

$$= \frac{e^{-\lambda q t}(\lambda q t)^n}{n!}$$

$\blacksquare$

5. Cell phone calls handled by a tower occur according to a Poisson process with parameter λ. Each call lasts for a period of time X having distribution G. Find the distribution of the number of calls being handled by the tower at time T.

(Assume no ongoing calls at time 0.)

Solution Sketch

A call arriving at s will still be active at T if $X > T-s$, that a call arriving at s will be active at T with probability

$$p(s) = P(X > T-s) = \bar{G}(s).$$

Now we see that the number of calls being handled at T is the Type I process described in Problem 4. Thus, no. of calls being handled at time $T \sim \text{Poisson}\left(\lambda \int_0^T \bar{G}(s)\, ds\right)$.

$\blacksquare$